Quantum Optics I: Quasiprobability Distribution and Quantum State reconstruction.
– Lecture 4 –

P-Function Glauber Sudarshan Phase Space Fct.

Assume we have normally ordered operator $\hat{O} = :f(\hat{a}, \hat{a}^{+}):$
e.g. $f = \hat{a}^{+}\hat{a}\hat{a}$

Thus $\langle \alpha | :f(\hat{a}, \hat{a}^{+}): | \alpha \rangle = \underbrace{f(\alpha, \alpha^{*})}_{\text{c-number function}}$ i.e. replace $a \to \alpha$, $\hat{a}^{+} \to \alpha^{*}$

Simplified density operator: which is diagonal in coherent state representation

$$\boxed{\rho = \int \underbrace{P(\alpha)}_{\text{Glauber-Sudarshan phase space function}} |\alpha\rangle\langle\alpha| \, \partial^2\alpha}$$

$P(\alpha)^{*} = P(\alpha)$ real valued

Properties: $\int \partial^2\alpha \, P(\alpha) = 1 = \mathrm{tr}(\rho)$ note $P(\alpha) = P(\alpha, \alpha^{*}) = P(\alpha_R, \alpha_{imag})$

This function can be used to compute expectations:

$$\boxed{\langle \hat{a}^{+m} \hat{a}^{n} \rangle = \mathrm{Tr}\{ \rho (\hat{a}^{+})^{m} (\hat{a})^{n} \} = \int d^2\alpha \cdot P(\alpha, \alpha^{*}) . (\alpha^{*})^{m} (\alpha)^{n}}$$

→ i.e. very simple operator correspondence. Thus $P(\alpha, \alpha^{*})$ is a quasi-probability function.

i.e. P diagonalizes the density operator in the coherent state basis.

Examples of the representation:

Coherent state: $\rho = |\alpha_0\rangle\langle\alpha_0|$

then $\boxed{P(\alpha) = \delta(\alpha - \alpha_0) \text{ trivial case}}$

variance:
$$\langle \Delta\hat{u}^2 \rangle = \langle (\hat{a}^{+}\hat{a})^2 - \langle \hat{a}^{+}a \rangle^2 \rangle$$
$$= \underbrace{\langle a^{+2}\hat{a}^2 + a^{+}a - \langle \hat{a}^{+}a \rangle^2 \rangle}_{\text{normal ordering}}$$

$$\langle \Delta\hat{u}^2 \rangle = \int \partial^2\alpha \, \underbrace{P(\alpha, \alpha^{*})}_{\delta(\alpha - \alpha_0)} (|\alpha|^4 + |\alpha|^2) - \left[\int \partial^2\alpha \, \underbrace{P(\alpha, \alpha^{*})}_{\delta(\alpha - \alpha_0)} |\alpha|^2 \right]^2$$
$$\hat{=} |\alpha_0|^2$$

Quantum Optics Lecture 4
(Phase Sp. Distributions)

Example Fock state $|n\rangle$ (i.e. Number State) P-representation.

$\rho = |n\rangle\langle n|$

(without proof)

$$\boxed{P(\alpha,\alpha^*) = \cancel{\ldots} \; \frac{\partial^{k}}{\partial\alpha^{k}} \underbrace{\frac{\partial^{k}}{\partial\alpha^{*k}} \delta^2(\alpha)}_{\text{k-th derivative of } \delta\text{-function}}}$$

$\delta^2\alpha = \partial(\alpha_{re})\,\partial(\alpha_{im})$

→ highly singular Function and can become NEGATIVE. But valid to compute normally ordered correlations.

Thermal state $\boxed{P(\alpha) = \frac{1}{\pi n_{th}} \exp\left(\frac{-|\alpha|^2}{n_{th}}\right)}$ well behaved.

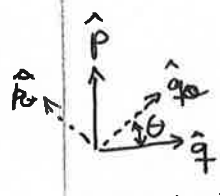
WIGNER REPRESENTATION

The Wigner function is used to obtain expectation values of SYMMETRICALLY ORDERED quantities ie operators

One example is homodyne detection which yields spectra of the form $\propto (\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$ i.e $(2a^\dagger a + \underbrace{1}_{\text{quantum noise}})$.

$$\langle f(\hat{a},\hat{a}^\dagger)\rangle = \langle \underbrace{\hat{f}^{(s)}(\hat{a},\hat{a}^\dagger)}_{\text{symmetric order}}\rangle = \int P^{(s)}(\alpha,\alpha^*)\, d^2\alpha\, f^{s}(\alpha,\alpha^*)$$

and $\boxed{P^{(s)}(\alpha,\alpha^*) \equiv W(\alpha) \quad : \text{Wigner function}}$



rotated basis

see also later! p.6)

* Historical remark 1932 Wigner $W(q,p)$ *

$P(q,\theta) = \langle q_\theta|\rho|q_\theta\rangle$ and $(= \langle q|U_\theta^\dagger \hat{\rho} U_\theta|q\rangle)$

$q_\theta = q\cos\theta + p\cos\theta$

$p_\theta = q\sin\theta - p\cos\theta$

The marginals reproduce the distributions of $\hat{q}$ ∧ $\hat{p}$ for

MOMENTUM DISTRIBUTION / PROBABILITIES — POSITION DISTRIBUTION:

$\langle q_{\pi/2}|\rho|q_{\pi/2}\rangle = \psi(p) = \int W(q,p)\,dq$ and $\psi(q) = \int W(q,p)\,dp = \langle q|\hat{\rho}|q\rangle$ MARGINALS*

Wigner: $\boxed{W(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ipx)\langle q - \frac{x}{2}|\hat{\rho}|q + x/2\rangle\,dx}$ Wigner-Weyl distribution

→ Proof: Ch 3.1 Ulf Leonhardt.

*remarkable feature links $W(q,p)$ to observations.

Basic properties (i) $W^*(q,p) = W(q,p)$ is real

(ii) Normalized $\iint W(q,p)\,dq\,dq = 1$

(iii) Overlap $\mathrm{Tr}\{\hat{\rho}\hat{F}\} = 2\pi \iint W(q,p)\, W_F(q,p)\,dq\,dp$

or $\langle\psi_1|\psi_2\rangle = \int W_1(q,p)\, W_2(q,p)\,dq\,dp$

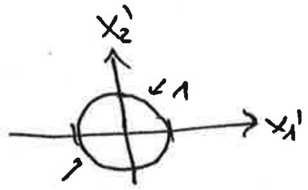
(iv) Purity $\mathrm{Tr}(\rho^2) \leq 1 = \int 2\pi\, W(q,p)^2\,dq\,dp$

Quantum Optics Lecture 3
(Phase space distribution tct.)

Coherent state $\quad |\alpha\rangle = |\tfrac{1}{2}\overline{X_1} + \tfrac{1}{2}\overline{X_2}i\rangle$

$$W(x_1', x_2') = \frac{2}{\pi} e^{-\frac{1}{2}(x_1'^2 + x_2'^2)}$$

$x_1' = X_1 - \overline{X_1}$ and $x_1'^2 + x_2'^2 = 1$ contour

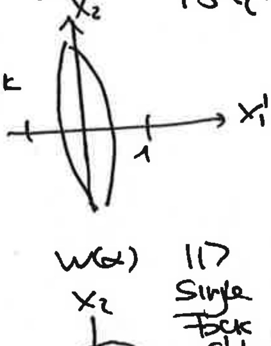


$\xi = re^{i\varphi}$

since $\left| \begin{array}{l} S^+(\xi)\hat{X}_1 S(\xi) = e^{\mp r}\hat{X}_1 \\ S^+(\xi)\hat{X}_2 S(\xi) = e^{\pm r}\hat{X}_2 \end{array} \right|$ $\varphi=0/\pi$

squeezed state

$$W(x_1', x_2') = \frac{2}{\pi} e^{-\frac{1}{2}(x_1'^2 e^{-2r} + x_2'^2 e^{+2r})}$$

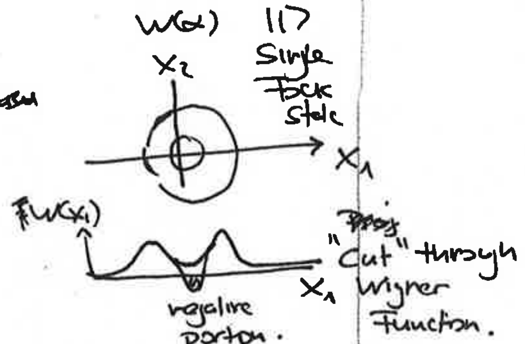


number state: $|n\rangle$ $\quad \underbrace{\sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!\,k!} \frac{n!}{k!}(2\alpha)^{2k}}$

$$W(x_1, x_2) = \frac{2}{\pi}(-1)^n \cdot L_n(4r^2)\, e^{-2r^2}$$

L_n: Laguerre Polynomials

$\rightarrow$ WIGNER FCT is NEGATIVE



Wigner function for one photon Fock state $|1\rangle$: (no proof, cf. Charmichael)

$$\left| W(\alpha, \alpha^*) = \frac{2}{\pi} e^{-2|\alpha|^2} (4|\alpha|^2 - 1) \right|$$

$\rightarrow$ is negative at origin. $|\alpha| < 1/2$.

example: compute

$$\frac{1}{2}(a^+a + aa^+) = \frac{1}{2}(2a^+a + 1) = \frac{1}{2}(2n+1)$$

verify:

$$\langle \alpha^* \alpha \rangle_S \equiv \int d^2\alpha\, W(\alpha, \alpha^*)\, \alpha^* \alpha \qquad \text{(cf. Charmichael 4.2)}$$

$$\hat{=} \frac{1}{2}(2n+1)$$

HISTORICAL NOTE (cf. LEON COHEN "Time-freq distr. review")

SIDE REMARK: joint Time-frequency probability functions from Wigner have also been used in classical signal analysis: $s(t)$ signal

$$W(t,\omega) = \frac{1}{2\pi} \int s^*(t - \tfrac{1}{2}\tau)\, e^{-i\omega\tau}\, s(t + \tfrac{1}{2}\tau)\, d\tau \qquad \text{WIGNER DE VILLE FCT}$$

VILLE used WIGNER's QM DISTRIBUTION

$\Downarrow$ satisfies joint time-distribution

$$\int_{-\infty}^{\infty} W(t,\omega)\, d\omega \propto |s(t)|^2 \quad \text{signal}$$

$$\int_{-\infty}^{\infty} W(t,\omega)\, dt \propto |S(\omega)|^2 \quad \text{spectrum}$$

Quantum Optics I

Quasi-probability distributions

Ref: Charmichael Stoch. Methods

P-representation

Q: Can a $P(\alpha,\alpha^*)$ be found for every $S_{n,m}$ density matrix?

Note: $$\underbrace{\mathrm{Tr}\{\rho e^{iz^*a^+} e^{iza}\}}_{\text{"characteristic function" } \chi_N(z,z^*)} = \mathrm{Tr}\{\int d^2\alpha |\alpha\rangle\langle\alpha| \overbrace{P(\alpha)}^{\text{assume P exists.}} e^{iz^*\hat{a}^+} e^{iz\hat{a}}\}$$

$$= \int d^2\alpha\, P(\alpha,\alpha^*) \langle\alpha| e^{iz^*a^+} e^{iza} |\alpha\rangle$$

assume diagonal ie $\alpha \neq \beta$ vanishes

$$= \int d^2\alpha\, P(\alpha,\alpha^*) e^{iz^*\alpha^*} e^{iz\alpha} \quad \text{(complex integral)}$$

→ This is a 2D Fourier transform (!) Invert: $dt e^{+i\omega t} \to e^{iz^*\alpha} d\alpha$; $d\omega e^{-i\omega t} \to e^{-iz\alpha} dz$

$$P(\alpha) = \frac{1}{\pi^2}\int d^2z\, \mathrm{tr}\{\rho e^{iz\hat{a}^+} e^{iz\hat{a}}\} e^{-iz^*\alpha^*} e^{-iz\alpha} \;\hat{=}\; \frac{1}{\pi^2}\int d^2z\, \chi_N(z,z^*) e^{-iz^*\alpha^*} e^{-iz\alpha}$$

Inverse transformation in $\partial^2 z = \partial z_{RE}\, \partial z_{imy}$. use Fock Basis. (complicated exp's. see Charmichael)

$$p(\alpha) = \int d^2z \sum_n\sum_m\sum_k S_{n+k,m+k} \cdot \frac{\sqrt{(n+k)!}\sqrt{(m+k)!}}{k!} \frac{(iz^*)^n}{n!} \frac{(iz)^m}{m!} e^{-iz^*\alpha^*} e^{iz\alpha}$$

P-Function expressed as $S_{n,m}$: Fock state Basis

Example: Fock state $|n\rangle$

$$P(\alpha) = \underbrace{\delta^2\alpha}_{\delta(\alpha_{RE})\delta(\alpha_{imy})\ (\text{DIRAC})} + \frac{\partial^2}{\partial\alpha\,\partial\alpha^*} \underbrace{\delta^2\alpha}_{\text{DIRAC-}\delta}$$

→ Problem for graphical representation and measurement!

ie derivatives of δ-function (not well behaved). Compare to the WIGNER FCT. which is well behaved

The characteristic function defines all NORMAL ORDERED operator products

$$\chi_N(z,z^*) \equiv \mathrm{Tr}\{\rho e^{iz^*\hat{a}^+} e^{iz\hat{a}}\}$$

$$\langle \hat{a}^{+p}\hat{a}^q\rangle = \mathrm{tr}\{\rho \hat{a}^{+p}\hat{a}^q\} \;\hat{=}\; \frac{\partial^{p+q}}{\partial(iz^*)^p\,\partial iz^q} \chi_N(z,z^*)\Big|_{z=z^*=0}$$

Note that from this also follows the expected relation

$$\langle \hat{a}^{+p}\hat{a}^q\rangle = \frac{\partial^{p+q}}{\partial^p(iz^*)\,\partial^q(iz)} \Big(\underbrace{\int d^2\alpha\, P(\alpha,\alpha^*) e^{iz^*\alpha^*} e^{iz\alpha}}_{\chi_N(z,z^*)}\Big)\Big|_{z=z^*=0}$$

$$\hat{=} \quad = \int d^2\alpha\, \alpha^q \alpha^{*p}\, P(\alpha,\alpha^*)$$

i.e as before.

Note: The P-function is ill behaved for e.g Fock states. It is well suited to compute e.g correlations e.g $g^1(T)$, $g^{(2)}(T)$

$$g^2(T) = \langle \hat{a}^+_k(t)\hat{a}^+_k(t+T)\hat{a}_k(t+T)\hat{a}_k(t)\rangle$$ but not homodyne

→ Homework: Bunching & anti-bunching.

(Ref. Charmichael)

Quantum Optics I
- Lecture 4 -

Quasi-Probability Fct: The WIGNER distribution Function & Q-Function

Symmetrically ordered Fcts $(\hat{a}^\dagger \hat{a})_S \equiv (\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger)\frac{1}{2}$

anti-symmetric: $(\hat{a}^p \hat{a}^{+q})$ $\qquad (a^{+2}a)_S \equiv \frac{1}{3}(a^{+2}a + a^\dagger a a^\dagger + a a^{+2})$ $\vdots$

First anti-symmetric order (Q-Fct) HUSIMI-Q. Fct.

$$\chi_A \equiv \mathrm{tr}(\rho\, e^{iz\hat{a}} e^{iz^*\hat{a}^*})$$

$$\langle \hat{a}^q (\hat{a}^+)^p \rangle = \mathrm{tr}(\rho\, \hat{a}^q \hat{a}^{+q}) = \left.\frac{\partial^{p+q}}{\partial(iz^*)^p \partial^q (iz)} \chi_A(z,z^*)\right|_{z=z^*=0}$$

as before:

$$\langle \hat{a}^q (a^\dagger)^p \rangle = \int \partial^2\alpha\, Q(\alpha,\alpha^*)\, \alpha^{*p} \alpha^q$$

$Q(\alpha,\alpha^*) = \frac{1}{\pi}\langle \alpha | \rho | \alpha \rangle$ $\Rightarrow$ i.e. prob to observe a coherent state

$$\hat{=} \frac{1}{\pi}\int d^2\beta\, e^{-|\beta-\alpha|^2} P(\beta,\beta^*)$$

Fock state $|n\rangle$

$$Q_{|n\rangle}(\alpha,\alpha^*) = \langle \alpha | n \rangle \langle n | \alpha \rangle \frac{1}{\pi} = \frac{|\langle \alpha | n \rangle|^2}{\pi} = \frac{e^{-|\alpha|^2/2}}{\pi}\frac{\alpha^n}{\sqrt{n!}}$$

WIGNER FUNCTION:

$$\chi_S \equiv \mathrm{tr}(\rho\, e^{iz^*\hat{a}^+ + iza})$$ characteristic Fct.

allows computing symmetric products of operators. $(a^\dagger a)_S = (\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$ SYMMETRIZED

$$W(\alpha,\alpha^*) \equiv \int d^2z\, \chi_S(z,z^*)\, e^{-iz^*\alpha^*} e^{-iz\alpha}$$ WIGNER FUNCTION

The Wigner function is the FOURIER TRANSFORM of $\chi_S(z,z^*)$

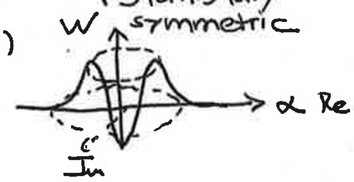
$$\langle \hat{a}^{+q} a^q \rangle_S = \int \partial^2\alpha\, W(\alpha,\alpha^*)(\alpha^*)^p \alpha^q$$

The Wigner function plays, as we will see, a key role in HOMODYNE DETECTION and QUANTUM STATE RECONSTRUCTION. IT IS NEGATIVE FOR NONCLASSICAL STATES

Wigner Function of a Fock State: (no proof) $|n\rangle = |1\rangle$

$$W(\alpha,\alpha^*) = \frac{2e^{-2|\alpha|^2}}{\pi}(4|\alpha|^2 - 1)$$

$W(\alpha,\alpha^*)$



→ Rotationally symmetric (phase is undefined)

Relation to $P(\alpha,\alpha^*)$ (no proof)

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\int_{-\infty}^{\infty} \partial^2\beta\, e^{-2|\alpha-\beta|^2} P(\beta,\beta^*)$$ i.e. averaged P function.

Quantum Optics I
Lecture 4

MEASUREMENT

Homodyne: $X_\varphi = X_1 \cos(\varphi) + X_2 \sin(\varphi)$

$\langle X_\varphi \rangle$ – measurements N

Pr X_φ "binning" MARGINALS

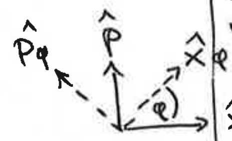
Det. Ice $\hat{X}_\varphi$ $|\xi\rangle \to$ $|\alpha\rangle$

NOTE

Quantum State Tomography (continuous variable Q state tomography)

Wigner function: $W(\alpha,\alpha^*) = W(q,p)$

Quadrature distribution
$$\boxed{\begin{aligned}\hat{X}_\varphi &= \hat{x}\cos\varphi + \hat{p}\sin\varphi \\ \hat{P}_\varphi &= -\hat{x}\sin\varphi + p\cos(\varphi)\end{aligned}}$$



$\hat{P}_\varphi = \hat{X}_{\varphi+\pi/2}$

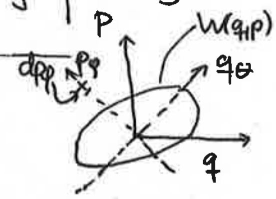
we need to compute the distribution (Prob-density function) for making a measurement with outcome X_φ

$$[*]\quad \underbrace{P_\Phi(X_\varphi)}_{\text{Probability distribution of } X_\varphi} = \int_{-\infty}^{\infty} W(\underbrace{X_\varphi\cos\varphi - P_\Phi\sin\varphi}_{\hat{x}^\varphi},\ \underbrace{X_\varphi\sin\Phi + P_\Phi\cos\varphi}_{\hat{p}})\,dp_\varphi$$

X_φ: observable (outcome of measurement)

MARGINAL

* integration along "cut"



Note that $\langle \hat{X}_\varphi \rangle$ is expectation value is NOT sufficient to characterize the quantum state. We need the distribution.

[*] key relationship between Wigner Function and homodyne signals

* Expression is called MARGINAL and gives prob. distribution

ie $\left| \underbrace{|\langle x_\varphi | \hat{X}_\varphi | x_\varphi\rangle|^2}_{\text{expectation value with } |x_\varphi\rangle} \hat{=} P_\varphi(X_\varphi) \right| \hat{=} \langle X_\varphi | \hat{\varsigma} | \underbrace{X_\varphi}_{\text{Quadrature Eigenstate}} \rangle$ with X_φ

Note:

i.e $\hat{X}_\varphi |X_\varphi\rangle = X_\varphi |X_\varphi\rangle$ is an eigenstate.

* Original Wigner Function (Wigner 1932) *

$$\boxed{W_{\hat{\varsigma}}(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle q + \tfrac{1}{2}q' | \hat{\varsigma} | q - \tfrac{1}{2}q' \rangle e^{-ipq'}\,dq'}$$

HISTORICAL DEFINITION 1932 WIGNER

$W^*(q,p) = W(q,p)$: real valued function.

W(q,p) is a joint probability function for $\hat{p},\hat{q}$ operator measurement outcomes. i.e

$$pr(q) = \langle \underbrace{q}_{\text{eigenstate of position}} | \underbrace{\hat{\varsigma}}_{\text{density matrix}} | q \rangle = \int_{-\infty}^{\infty} W(q,p)\,dp$$

likewise (ie $Pr(q,\theta=\pi/2 \hat{=} Pr(p)$):

$$pr(q,\theta=\pi/2) = pr(p) = \langle \underbrace{p}_{\text{momentum Eigenstate}} | \hat{\varsigma} | p \rangle = \int_{-\infty}^{\infty} W(q,p)\,dq$$

MARGINALS.

WIGNER FUNCTIONS GIVES MOMENTUM & POSITION PROB. DISTRIBUTION (as before)

NOTE: the definition of MARGINALS defines WIGNER FCT

For arbitrary rotation

$$\boxed{pr(q,\theta) = pr(q_\theta) \hat{=} pr(X_\varphi) = \int_{-\infty}^{\infty} W(q\cos\theta - p\sin\theta,\ q\sin\theta + p\cos\theta)\,dp}$$

$\hat{=} \langle q | U(\theta) \hat{\varsigma} U^\dagger(\theta) | q \rangle$

Quantum Optics I
– Lecture 9.

Quantum State TOMOGRAPHY (7.1)

Refs: Vogel) Introduction to QO
Ulf LEONHARDT measuring the quantum state of light

Motivation: How to reconstruct a quantum state of light e.g. a squeezed state? $\rho_{nm} = \langle n|\rho|m\rangle$

$\langle \hat{I}_{\text{photodetector}}\rangle \langle a^\dagger a\rangle = \mathrm{Tr}\{\rho a^\dagger a\}$
$\sum \rho_{nn} \cdot n$

- Issue we cannot directly measure e.g. $\rho_{nn} = \langle n|\rho|n\rangle$ with a photodetector, i.e. "Photon resolving" measurements. Exception: (later) CQED in dispersive regime allows measuring $p_n = \rho_{nn}$ → Photon number statistics or QND in CQED → Haroche experiment $\rho_{n,m}$

Key point: Measurement of $P_{\Phi}(x_\varphi)$ provides way to reconstruct $W(q,p)$ by noting that

$$\boxed{P_{\Phi}(x_\varphi) = \int_{-\infty}^{\infty} W(q,p)\,dp}$$

(marginal along axis $\hat{x}_\varphi$, $\hat{x}_{\varphi+\pi/2}$)

one can use inverse RADON transform to reconstruct the state (for rotationally symmetric states e.g. Fock states it becomes simple Abel transform)

[Nobel '79 Makrve]

$$P(x_\varphi) = \int_{-\infty}^{\infty} W(q_\varphi \cos\varphi - p_\varphi \sin\varphi, \sin\varphi\, q_\varphi + \cos(\varphi) p)\, dp_\varphi$$

integral along a "cut" of wigner function $W(q,p)$

→ Inverse transformation Radon 1917.

For rotationally symmetric state in p,q space:

$$\boxed{W(r) = -\frac{1}{\pi}\int_r^{\infty} pr(x_\varphi)\,(x_\varphi^2 - r^2)^{-1/2}\,dx_\varphi \quad (x_\varphi = \text{const.})}$$

Important comment: imperfect measurements will destroy reconstruction as vacuum state get's added as statistical mixture

Note: also the density matrix $\rho_{n,m}$ can be reconstructed from the wigner function.

$$\boxed{\rho_{n,m} = \int \underbrace{W(q,p)}_{\substack{\text{wigner}\\\text{function}\\\text{of state}}} \cdot \overbrace{\underbrace{W_{nm}(q,p)}_{\substack{\text{representation}\\\text{of } \rho = |n\rangle\langle m| \text{ projector}}}}^{\text{Pattern Function}}\,dp\,dq}$$

Quantum Optics I
- Lecture 4 -

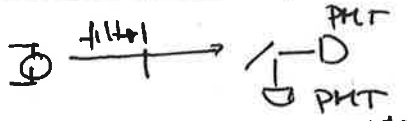
Ref: Scully Q.O.
Milburn, Q.O

$\boxed{\text{" correlation between photons in two coherent beams of light" HBT 1956}}$

→ homework

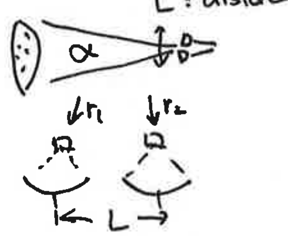
Experiment

Arclamp filter PMT PMT



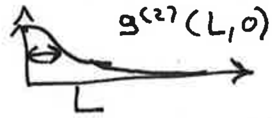
Spatial coherence

L : distance



$g_2(r_1, r_2, \tau=0)$

$g^{(2)}(L,0)$



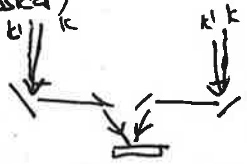
$\alpha = \lambda / L_c$
→ spatial coherence

Correlation (2nd)
$g^{(2)}(r_1, r_2, \tau)$

$$g^{(2)}(L,0) = \frac{\langle I(r_1,t), I(r_1+L, t+\tau)\rangle}{\langle I(r_1,t)\rangle \langle I(\underbrace{r_2}_{r_1+L},t)\rangle}$$

Contrast to Michelson Interferometer (classical)

$$g^{(1)} = \frac{\langle E^*(r_1,t) E(r_2,t+\tau)\rangle}{\langle |E(r_1,t)|^2\rangle^{1/2} \langle |E(r_2,t+\tau)|^2\rangle^{1/2}}$$



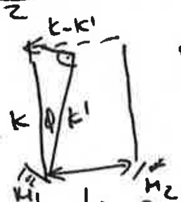
First order coherence

First order coherence.
→ sensitive to path length:
$$\propto \cos\left((k+k')\underbrace{(r_1' - r_2)}_{\text{Path length difference}}\right) \cdot \cos\left(\frac{\pi r \alpha}{\lambda}\right)$$

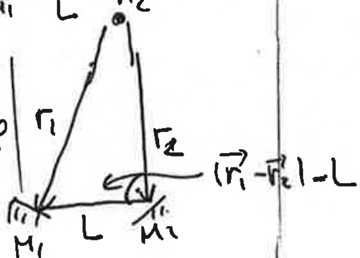
Assume that $|E_k|^2 = |E_{k'}|^2 \approx I_0$

$$I \propto \langle E^* E\rangle = I_0 \{ 1 + \cos\left((k+k')\frac{(r_1 - r_2)}{2}\right)\cos\left(\underbrace{\frac{(k-k')(r^1 - r^2)}{2}}_{\propto \varphi k}\right)$$

→ one can determine distance to star / galaxy. Problem: atmosphere in earth.



$$\boxed{\begin{aligned}(\vec{k} - \vec{k}')(\vec{r}_1 - \vec{r}_2) &= \underbrace{|\vec{k} - \vec{k}'|}_{|k - k'| \approx k\cdot\varphi = k\sin\varphi}\underbrace{\cos\varphi}_{\approx 1} L \\ &= k \cdot \varphi \cdot L\end{aligned}}$$



Return to 2nd order: (equal time).

Note: interference present!

$$\langle I(r_1,t) \cdot I(r_2,t)\rangle = \left(|E_k|^2 + |E_{k'}|^2 + E_k E_{k'}^* e^{-i(k-k')r_2} + c.c.\right) \times \left(|E_k|^2 + |E_{k'}|^2 + E_k E_{k'}^* e^{-i(k-k')r} + cc\right)$$

$$= \left(|E_{k'}|^2 + |E_k|^2\right)^2 + |E_k|^2 |E_{k'}|^2 \left(e^{-i(\vec{k}-\vec{k}')(\vec{r}_1 - \vec{r}_2)} + cc\right)$$

→ no term such as $\cos\left((k+k')\frac{(r_1 - r_2)}{2}\right)$ → fast oscillations (λ-scale)

But: $e^{(i(k-k')(\vec{r}_1 - \vec{r}_2)} \propto$ ~~[illegible]~~ $\varphi\left(\frac{2\pi}{\lambda}\right) \cdot L$